Fields of definition of building blocks with quaternionic multiplication

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Abstract

This paper investigates the fields of definition up to isogeny of the abelian varieties called *building blocks*. In [5] and [3] a characterization of the fields of definition of these varieties together with their endomorphisms is given in terms of a Galois cohomology class canonically attached to them. However, when the building blocks have quaternionic multiplication, then the field of definition of the varieties can be strictly smaller than the field of definition of their endomorphisms. What we do is to give a characterization of the field of definition of the varieties in this case (also in terms of their associated Galois cohomology class), by translating the problem into the language of group extensions with non-abelian kernel. We also make the computations that are needed in order to calculate in practice these fields from our characterization.

Contents

1 Introduction

An abelian variety $B/\overline{\mathbb{Q}}$ is called a $\mathbb{Q}$-*abelian variety* if for each $\sigma \in G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ there exists an isogeny $\mu_\sigma\colon {}^\sigma B \to B$ *compatible* with the endomorphisms of B, i.e. such that $\varphi \circ \mu_\sigma = \mu_\sigma \circ {}^\sigma\varphi$ for all $\varphi \in \mathrm{End}^0(B) = \mathrm{End}(B) \otimes_{\mathbb{Z}} \mathbb{Q}$. A *building block* is a $\mathbb{Q}$-abelian variety B whose endomorphism algebra $\mathrm{End}^0(B)$ is a central division algebra over a totally real number field F with Schur index $t = 1$ or $t = 2$ and $t[F : \mathbb{Q}] = \dim B$. In the case $t = 2$ the quaternion algebra is necessarily totally indefinite. The interest in the study of the building blocks comes from the fact that they are the absolutely simple factors up to isogeny of the non-CM abelian varieties of GL_2-type

(see [3]) and therefore, as a consequence of a generalization of Shimura-Taniyama, they are the non-CM absolutely simple factors of the modular jacobians $J_1(N)$.

In [5] and in [3] Ribet and Pyle investigated the possible fields of definition of a building block up to isogeny; in fact, and to be more precise, their results concern the field of definition of the variety together with its endomorphisms. The main result in this direction is that every building block $B/\overline{\mathbb{Q}}$ is isogenous over $\overline{\mathbb{Q}}$ to a variety B_0 defined over a polyquadratic[1] number field K, and with all the endomorphisms of B_0 also defined over K (this is [3, Theorem 5.1]). From the proof of this result one can deduce the existence of minimal polyquadratic number fields with this property. Moreover, each of these minimal number fields must contain a certain field K_P that can be calculated from a cohomology class in $H^2(G_{\mathbb{Q}}, F^*)$ canonically attached to B.

If B is a building block whose endomorphism algebra $\mathrm{End}^0(B)$ is a number field F and B is defined over a number field K, then all the endomorphisms of B are also defined over K; this follows easily from the compatibility of the isogenies and from the commutativity of $\mathrm{End}^0(B)$. Therefore, in this case it is not a restriction to impose on a field of definition of B to be also a field of definition of its endomorphisms. But if B has quaternionic multiplication, that is if $\mathrm{End}^0(B)$ is a quaternion algebra, then a field of definition of B is not necessarily a field of definition of $\mathrm{End}^0(B)$. In this situation, it can occur that B is indeed isogenous to a variety B_0 defined over a field L smaller than the minimal ones given by Ribet and Pyle, but of course with $\mathrm{End}^0_L(B_0) \subsetneq \mathrm{End}^0(B_0)$. The easiest case where this happens is in the abelian varieties of GL_2-type that are absolutely simple and have quaternionic multiplication over $\overline{\mathbb{Q}}$. They are building blocks and any field of definition of their endomorphisms must strictly contain $\mathbb{Q}$, but clearly $\mathbb{Q}$ can be taken to be a field of definition of these varieties up to isogeny. In section 6 we will give more involved examples of this phenomenon, in the sense that it will not be obvious a priori that one can descent the field of definition of the building block up to isogeny.

The goal of this article is to characterize the fields of definition of quaternionic building blocks up to isogeny, and to determine under what conditions it is possible to define them in a field strictly contained in the minimal ones given by Ribet and Pyle for the variety and the endomorphisms. Our strategy will be to translate the problem into the language of group extensions, and then use general results of this theory and some explicit computations with the groups involved to perform the study.

2 Building blocks and fields of definition

We begin this section by recalling the main tools used in the study of the field of definition of building blocks. The main references for this part are [5] and [3] (and see also [4, Section 1] for a similar account of this material).

Let K be a number field. We will say that a building block B is *defined over* K if the variety B (but not necessarily all of its endomorphisms) is defined over K. If B is isogenous to a building block defined over K we will say that K is *a field of definition of* B *up to isogeny*, or that B is *defined over* K *up to isogeny.* Note that this is a modification of the terminology used in [3], where a field of definition of a building block was defined to be a field of definition of the variety and of all its endomorphisms.

Given a building block B we fix for every $\sigma \in G_{\mathbb{Q}}$ a compatible isogeny $\mu_\sigma \colon {}^\sigma B \to B$. Since B

[1] i.e. a composition of quadratic extensions of $\mathbb{Q}$.

has a model defined over a number field, we can choose the collection $\{\mu_\sigma\}$ to be locally constant. For $\sigma,\tau\in G_{\mathbb{Q}}$ the isogeny $c_B(\sigma,\tau)=\mu_\sigma\circ{}^\sigma\mu_\tau\circ\mu_{\sigma\tau}^{-1}$ lies in the center F of $\mathrm{End}^0(B)$, and the map $(\sigma,\tau)\mapsto c_B(\sigma,\tau)$ is a continuous 2-cocycle of $G_{\mathbb{Q}}$ with values in F^* (equipped with the trivial $G_{\mathbb{Q}}$-action). Its cohomology class $[c_B]$ is an element of $H^2(G_{\mathbb{Q}},F^*)$ that does not depend on the particular choice of the compatible isogenies μ_σ, and if $B\sim B'$ are isogenous building blocks then we can identify its associated cohomology classes $[c_B]$ and $[c_{B'}]$. An important property of $[c_B]$ is that it belongs to the 2-torsion subgroup $H^2(G_{\mathbb{Q}},F^*)[2]$; that is, there exists a continuous map $\sigma\mapsto d_\sigma\colon G_{\mathbb{Q}}\to F^*$ such that $c(\sigma,\tau)^2=d_\sigma d_\tau d_{\sigma\tau}^{-1}$. The cohomology class $[c_B]$ gives all the information about the field of definition of a building block together with its endomorphisms up to isogeny, thanks to the following proposition, which is [3, Proposition 5.2].

Theorem 2.1 (Ribet-Pyle) *Let B be a building block and $\gamma=[c_B]$ its associated cohomology class. There exists a variety B_0 defined over a number field K and with all its endomorphisms defined over K that is isogenous to B if and only if $\mathrm{Res}_{\mathbb{Q}}^K(\gamma)=1$, where $\mathrm{Res}_{\mathbb{Q}}^K$ is the restriction map $\mathrm{Res}_{\mathbb{Q}}^K\colon H^2(G_{\mathbb{Q}},F^*)\to H^2(G_K,F^*)$.*

A *sign map* for F is a group homomorphism $\mathrm{sign}\colon F^*\to\{\pm1\}$ such that $\mathrm{sign}(-1)=-1$. A sign map gives a group isomorphism $F^*\simeq P\times\{\pm1\}$, where $P=F^*/\{\pm1\}$. From now on we fix a sign map for F by fixing an embedding of F in $\mathbb{R}$, and then taking the usual sign. The corresponding isomorphism $F^*\simeq P\times\{\pm1\}$ gives then a decomposition of $H^2(G_{\mathbb{Q}},F^*)[2]$.

Proposition 2.2 *Let F be a totally real number field, and let P be the group $F^*/\{\pm1\}$. There exists a (non-canonical) isomorphism of groups*

$$H^2(G_{\mathbb{Q}},F^*)[2]\simeq H^2(G_{\mathbb{Q}},\{\pm1\})\times\mathrm{Hom}(G_{\mathbb{Q}},P/P^2). \tag{1}$$

If $\gamma=[c]\in H^2(G_{\mathbb{Q}},F^*)[2]$ we denote by $\gamma_\pm\in H^2(G_{\mathbb{Q}},\{\pm1\})$ and $\overline{\gamma}\in\mathrm{Hom}(G_{\mathbb{Q}},P/P^2)$ its two components under the isomorphism (1). They can be computed in the following way:

1. *The cohomology class $\gamma_\pm$ is represented by the cocycle $(\sigma,\tau)\mapsto\mathrm{sign}(c(\sigma,\tau))$.*
2. *If $c(\sigma,\tau)^2=d_\sigma d_\tau d_{\sigma\tau}^{-1}$ is an expression of c^2 as a coboundary, the map $\overline{\gamma}$ is given by $\sigma\mapsto d_\sigma \bmod \{\pm1\}F^{*2}$.*

PROOF: This is essentially the content of the propositions 5.3 and 5.6 in [3]. □

Let B be a building block and $\gamma=[c_B]$ its associated cohomology class. A field K is a field of definition up to isogeny of B and of its endomorphisms if and only if K trivializes both components $\overline{\gamma}$ and $\gamma_\pm$ (that is, if and only if the restriction of both components to G_K is trivial). Let K_P be the fixed field of $\ker\overline{\gamma}$, which is a 2-extension of $\mathbb{Q}$. Then K trivializes $\overline{\gamma}$ if and only if it contains K_P. Since $H^2(G_{\mathbb{Q}},\{\pm1\})$ is isomorphic to the two torsion of the Brauer Group of $\mathbb{Q}$, we can identify $\gamma_\pm$ with a quaternion algebra over $\mathbb{Q}$, and K trivializes $\gamma_\pm$ if and only if it is a splitting field of the quaternion algebra represented by $\gamma_\pm$. If K_P already trivializes $\gamma_\pm$, then K_P is the minimum field of definition of B and of its endomorphisms up to isogeny. Otherwise, there is no such a minimum field: all the fields of definition of B and of its endomorphisms up to isogeny must contain K_P and are splitting fields of $\gamma_\pm$. For instance, for each maximal subfield $K_\pm$ of the quaternion algebra given by $\gamma_\pm$, the field $K_\pm K_P$ is a minimal polyquadratic number field with the property of being a field of definition of B and of its endomorphisms up to isogeny.

Our study of the fields of definition of a building block up to isogeny will be based on the following theorem of Ribet (cf. [6, Theorem 8.1]), that characterizes such fields.

Theorem 2.3 (Ribet) *Let L/K be a Galois extension of fields, and let B be an abelian variety defined over L. There exists an abelian variety B_0 defined over K such that B and B_0 are isogenous over L if and only if there exist isomorphisms in the category of abelian varieties up to isogeny $\{\phi_\sigma\colon {}^\sigma B \to B\}_{\sigma\in \mathrm{Gal}(L/K)}$ satisfying that $\phi_\sigma \circ {}^\sigma\phi_\tau \circ \phi_{\sigma\tau}^{-1} = 1$.*

Now we reformulate the previous theorem for building blocks in terms of the splitting of an exact sequence of groups. For a building block B, we denote by $\mathrm{Isog}^0({}^\sigma B, B)$ the isomorphisms between ${}^\sigma B$ and B in the category of abelian varieties up to isogeny. Let E be the disjoint union of these sets for all $\sigma \in G_K$:

$$E = \bigsqcup_{\sigma\in G_K} \mathrm{Isog}^0({}^\sigma B, B). \tag{2}$$

The elements in E are therefore of the form $\phi_\sigma \in \mathrm{Isog}^0({}^\sigma B, B)$ for some $\sigma \in G_K$. We note that, even if for some $\sigma \neq \tau$ one has ${}^\sigma B = {}^\tau B$, we regard $\mathrm{Isog}^0({}^\sigma B, B)$ and $\mathrm{Isog}^0({}^\tau B, B)$ as different sets in (2). We make E into a group by defining the product $\phi_\sigma \cdot \phi_\tau := \phi_\sigma \circ {}^\sigma\phi_\tau \in \mathrm{Isog}^0({}^{\sigma\tau} B, B)$. In fact, we will regard E as a topological group endowed with the discrete topology. If we denote by $\mathcal{B}$ the algebra $\mathrm{End}^0(B)$, for each number field K we have the following exact sequence

$$1 \longrightarrow \mathcal{B}^* \xrightarrow{\iota} E \xrightarrow{\pi} G_K \longrightarrow 1, \tag{3}$$

where ι consists on viewing every $\phi \in \mathcal{B}^*$ as an element of $\mathrm{Isog}^0({}^{\mathrm{Id}} B, B)$, and $\pi(\phi_\sigma) = \sigma$ if $\phi_\sigma \in \mathrm{Isog}^0({}^\sigma B, B)$. Note that π is surjective because B is isogenous to all of its Galois conjugates, since it is a $\mathbb{Q}$-variety.

Proposition 2.4 *Let B be a building block. A number field K is a field of definition of B up to isogeny if and only if the exact sequence* (3) *splits with a continuous section; that is, if and only if there exists a continuous morphism $s\colon G_K \to E$ such that $\pi \circ s = \mathrm{Id}$.*

Proof: A (set-theoretic) section for π is a map $\sigma \mapsto \phi_\sigma$ with $\phi_\sigma \in \mathrm{Isog}^0({}^\sigma B, B)$. By the definition of the product in E, the condition of being a morphism translates into the condition $\phi_\sigma \circ {}^\sigma\phi_\tau \circ \phi_{\sigma\tau}^{-1} = 1$, which is the same that appears in 2.3. □

3 Exact sequences with non-abelian kernel

The problem of determining whether an exact sequence of groups is split is well known and it has been vastly studied in the case of abelian kernel. However, we are interested in the sequence (3) only in the case that $\mathcal{B}$ is a quaternion algebra, and then the kernel is non-abelian. For this reason, in this section we consider exact sequences of topological groups

$$1 \longrightarrow H \xrightarrow{\iota} E \xrightarrow{\pi} G \longrightarrow 1, \tag{4}$$

where we do not require H to be commutative. Recall that a (set-theoretic) *section* of π is a continuous map $s\colon G \to E$ such that $\pi \circ s = \mathrm{Id}$, and that the sequence (4) is said to be *split* if there exists a section s that is also a group homomorphism. For each section s of π we define c_s to be the map

$$\begin{array}{rccc} c_s\colon & G\times G & \longrightarrow & H \\ & (\sigma,\tau) & \longmapsto & s(\sigma)s(\tau)s(\sigma\tau)^{-1}. \end{array}$$

Let Z be the center of H. We say that s is a *central section* if $c_s(\sigma,\tau) \in Z$ for all $\sigma, \tau \in G$. A central section s can be used to define an action Θ_s of G in H, by defining $\Theta_s(\sigma)(h) =$

$s(\sigma)hs(\sigma)^{-1}$ for all $\sigma \in G$, $h \in H$. It is easily checked that this is indeed a group action that will depend, in general, on the central section s used. However, when restricted to Z it gives rise to an action from G on Z that is independent of the central section s.

Let θ be the action defined on Z by any central section, and denote by $Z^2(G, Z; \theta)$ the group of continuous 2-cocycles from G with values in Z with respect to the action θ. One easily checks that $c_s \in Z^2(G, Z; \theta)$. In the commutative kernel case, i.e. when $H = Z$, the cohomology class $[c_s]$ does not depend on the section s, and the exact sequence (4) is split if and only if $[c_s]$ is trivial. However, if the kernel is non-abelian the cohomology class $[c_s]$ does depend in general on the central section s, and the characterization of the splitting of the exact sequence in terms of $[c_s]$ is given by the following

Proposition 3.1 *The exact sequence* (4) *is split if and only if there exists a central section s such that its cohomology class $[c_s] \in H^2(G, Z; \theta)$ is trivial.*

Proof: If the sequence splits there exists a section s that is also a group homomorphism, and therefore $c_s(\sigma, \tau) = s(\sigma)s(\tau)s(\sigma\tau)^{-1} = 1$. That is, s is a central section and $[c_s] = 1$.

Suppose now that s is a central section s with $[c_s] = 1$. This means that there exists a continuous map $\sigma \mapsto \eta_\sigma$ from G in Z such that

$$c_s(\sigma, \tau) = s(\sigma)s(\tau)s(\sigma\tau)^{-1} = \eta_\sigma\theta(\sigma)(\eta_\tau)\eta_{\sigma\tau}^{-1} = \eta_\sigma s(\sigma)\eta_\tau s(\sigma)^{-1}\eta_{\sigma\tau}^{-1} = \eta_{\sigma\tau}^{-1}s(\sigma)\eta_\tau s(\sigma)^{-1}\eta_\sigma.$$

Then, the map $t(\sigma) = \eta_\sigma^{-1}s(\sigma)$ is also a section and it is a group homomorphism:

$$\begin{aligned} t(\sigma)t(\tau)t(\sigma\tau)^{-1} &= \eta_\sigma^{-1}s(\sigma)\eta_\tau^{-1}s(\tau)s(\sigma\tau)^{-1}\eta_{\sigma\tau} = \eta_\sigma^{-1}s(\sigma)\eta_\tau^{-1}s(\sigma)^{-1}s(\sigma)s(\tau)s(\sigma\tau)^{-1}\eta_{\sigma\tau} = \\ &= \eta_\sigma^{-1}s(\sigma)\eta_\tau^{-1}s(\sigma)^{-1}\eta_{\sigma\tau}s(\sigma)s(\tau)s(\sigma\tau)^{-1} = \eta_\sigma^{-1}s(\sigma)\eta_\tau^{-1}s(\sigma)^{-1}\eta_{\sigma\tau}c(\sigma, \tau) = 1. \end{aligned}$$

$\square$

Hence, deciding whether an exact sequence is split is equivalent to deciding whether the set of all cohomology classes associated to central sections contains the trivial class. Now we show how to compute the set of all cohomology classes associated to central extensions from the knowledge of one particular class $[c_s]$ in this set.

Let s be a central section; we denote by Θ_s the action it defines on H and by θ the action it defines on Z. The following exact sequence of G-groups

$$1 \longrightarrow Z \longrightarrow H \longrightarrow H/Z \longrightarrow 1$$

gives rise to the cohomology exact sequence of pointed sets (cf. [7, p. 123])

$$1 \longrightarrow Z \longrightarrow H \longrightarrow H/Z \longrightarrow H^1(G, Z; \theta) \longrightarrow H^1(G, H; \Theta_s) \longrightarrow H^1(G, H/Z; \Theta_s) \xrightarrow{\delta} H^2(G, Z; \theta).$$

The explicit description of the connecting map δ is given in terms of cocycles by

$$\begin{array}{rlcl} \delta\colon & H^1(G, H/Z; \Theta_s) & \longrightarrow & H^2(G, Z; \theta) \\ & [\sigma \mapsto \psi_\sigma Z] & \longmapsto & [(\sigma, \tau) \mapsto \psi_\sigma\Theta_s(\sigma)(\psi_\tau)\psi_{\sigma\tau}^{-1}]. \end{array} \qquad (5)$$

Proposition 3.2 *The set of all cohomology classes associated to central sections is* $[c_s]\,\mathrm{im}(\delta) \subseteq H^2(G, Z; \theta)$.

Proof: Let $\psi\colon \sigma\mapsto\psi_\sigma Z$ be an element in $Z^1(G,H/Z;\Theta_s)$. Then

$$\begin{array}{rcl} c_s(\sigma,\tau)\delta(\psi)(\sigma,\tau) & = & c_s(\sigma,\tau)\psi_\sigma s(\sigma)\psi_\tau s(\sigma)^{-1}\psi_{\sigma\tau}^{-1} = \\ & = & c_s(\sigma,\tau)\psi_\sigma s(\sigma)\psi_\tau s(\tau)s(\sigma\tau)^{-1}s(\sigma\tau)s(\tau)^{-1}s(\sigma)^{-1}\psi_{\sigma\tau}^{-1} = \\ & = & c_s(\sigma,\tau)\psi_\sigma s(\sigma)\psi_\tau s(\tau)s(\sigma\tau)^{-1}c_s(\sigma,\tau)^{-1}\psi_{\sigma\tau}^{-1} = \\ & = & \psi_\sigma s(\sigma)\psi_\tau s(\tau)s(\sigma\tau)^{-1}\psi_{\sigma\tau}^{-1}, \end{array}$$

which is the 2-cocycle associated to the central section $\sigma\mapsto\psi_\sigma s(\sigma)$.

Let t be an arbitrary central section. For each $\sigma\in G$ we define $\psi_\sigma = t(\sigma)s(\sigma)^{-1}$. The map $\psi\colon \sigma\mapsto\psi_\sigma Z$ is continuous and we have that

$$\begin{array}{rcl} \delta(\psi)(\sigma,\tau) & = & \psi_\sigma s(\sigma)\psi_\tau s(\sigma)^{-1}\psi_{\sigma\tau}^{-1} = t(\sigma)s(\sigma)^{-1}s(\sigma)t(\tau)s(\tau)^{-1}s(\sigma)^{-1}s(\sigma\tau)t(\sigma\tau)^{-1} = \\ & = & t(\sigma)t(\tau)t(\sigma\tau)^{-1}t(\sigma\tau)s(\tau)^{-1}s(\sigma)^{-1}s(\sigma\tau)t(\sigma\tau)^{-1} = \\ & = & c_t(\sigma,\tau)t(\sigma\tau)s(\sigma\tau)^{-1}s(\sigma\tau)s(\tau)^{-1}s(\sigma)^{-1}s(\sigma\tau)t(\sigma\tau)^{-1} = \\ & = & c_t(\sigma,\tau)c_s(\sigma,\tau)^{-1}, \end{array}$$

and we see that $[c_t]\in[c_s]\,\mathrm{im}(\delta)$. $\square$

Proposition 3.1 together with 3.2 immediately give the following

Corollary 3.3 *The exact sequence* (4) *is split if and only if the following conditions hold:*

1. *There exists a central section s.*
2. *The set $[c_s]\,\mathrm{im}(\delta)\subseteq H^2(G,Z;\theta)$ contains the trivial cohomology class.*

Now we particularize these results to the exact sequence (3). Let B be a building block, $\mathcal{B}=\mathrm{End}^0(B)$, $F=Z(\mathcal{B})$ and $\{\mu_\sigma\colon {}^\sigma B\to B\}$ a locally constant set of compatible isogenies. Since the μ_σ are compatible the map $s\colon\sigma\mapsto\mu_\sigma$ is a central section for π, and the action it defines in $\mathcal{B}^*$ and in F^* is the trivial one. Therefore, with this action we can identify $H^1(G_K,\mathcal{B}^*/F^*)$ with $\mathrm{Hom}(G_K,\mathcal{B}^*/F^*)$, and the connection map δ is given by:

$$\begin{array}{rlcl} \delta\colon & \mathrm{Hom}(G_K,\mathcal{B}^*/F^*) & \longrightarrow & H^2(G,F^*) \\ & [\sigma\mapsto\psi_\sigma F^*] & \longmapsto & [(\sigma,\tau)\mapsto\psi_\sigma\psi_\tau\psi_{\sigma\tau}^{-1}]. \end{array} \tag{6}$$

Moreover, the cohomology class $[c_s]$ is just the restriction to G_K of $\gamma=[c_B]$. Putting all these considerations together we obtain, as an immediate consequence of 3.3, the following result.

Proposition 3.4 *Let B be a building block and $\gamma=[c_B]\in H^2(G_{\mathbb{Q}},F^*)$ its associated cohomology class. Then B is isogenous to a variety defined over a number field K if and only if there exists a continuous morphism $\psi: G_K\to\mathcal{B}^*/F^*$ such that $\delta(\psi)\,\mathrm{Res}_{\mathbb{Q}}^K(\gamma)=1$.*

Since γ is 2-torsion, the above proposition tells us that B is defined over K up to isogeny if and only if $\mathrm{Res}_{\mathbb{Q}}^K(\gamma)$ is equal to $\delta(\psi)$, for some morphism ψ. This result allows us to calculate fields of definition up to isogeny of building blocks by knowing the image of the connecting map δ. Therefore, before we continue with our study of fields of definition, we have to compute the image of δ.

4 The image of δ

This technical section is devoted to compute all the elements in $H^2(G_K, F^*)[2]$ that are of the form $\delta(\psi)$ for some continuous morphism $\psi\colon G_K \to \mathcal{B}^*/F^*$, and to determine their components $\delta(\psi)_\pm$ and $\overline{\delta(\psi)}$ under the isomorphism $H^2(G_K, F^*)[2] \simeq H^2(G_K, \{\pm 1\}) \times \mathrm{Hom}(G_K, P/P^2)$ (this isomorphism is just the restriction of (1) to G_K). The image of a continuous morphism $\psi\colon G_K \to \mathcal{B}^*/F^*$ is a finite subgroup of $\mathcal{B}^*/F^*$. In [1, Section 2] these subgroups are studied and, in particular, we have the following result:

Proposition 4.1 (Chinburg-Friedman) *Let $\mathcal{B}$ be a totally indefinite division quaternion algebra over a field F. The finite subgroups of $\mathcal{B}^*/F^*$ are cyclic or dihedral. There always exist subgroups of $\mathcal{B}^*/F^*$ isomorphic to C_2 and $C_2 \times C_2$. For $n > 2$, if ζ_n is a primitive n-th root of unity in $\overline{F}$, $\mathcal{B}^*/F^*$ contains a subgroup isomorphic to C_n if and only if $\zeta_n + \zeta_n^{-1} \in F$ and $F(\zeta_n)$ is isomorphic to a maximal subfield of $\mathcal{B}$. In this case, $\mathcal{B}^*/F^*$ always contains a subgroup isomorphic to a dihedral group D_{2n} of order $2n$.*

In order to compute the cohomology classes $\delta(\psi)$ we will consider four separate cases, depending on whether $\mathrm{im}\,\psi$ is isomorphic to C_2, $C_2 \times C_2$, C_n or D_{2n} for $n > 2$. The following notation may be useful: if G is a group, we denote by Δ_G the elements $\gamma \in H^2(G_K, F^*)[2]$ that are of the form $\gamma = \delta(\psi)$ for some morphism ψ with $\mathrm{im}\,\psi \simeq G$.

As usual we will identify the elements in $H^2(G_K, \{\pm 1\})$ with quaternion algebras over K, and we will use the notation $(a, b)_K$ for the quaternion algebra generated over K by i, j with $i^2 = a$, $j^2 = b$ and $ij + ji = 0$. As for the elements in $\mathrm{Hom}(G_K, P/P^2)$ we will use the notation $(t, d)_P$ with $t \in K$ and $d \in F^*$, to denote (the inflation of) the morphism that sends the non-trivial automorphism of $\mathrm{Gal}(K(\sqrt{t})/K)$ to the class of d in P/P^2. Every element in $\mathrm{Hom}(G_K, P/P^2)$ is the product of morphisms of this kind, and therefore it can be expressed in the form $(t_1, d_1)_P \cdot (t_2, d_2)_P \cdots (t_n, d_n)_P$ for some $t_i \in K$, $d_i \in F^*$. We remark that, although they are convenient for their compactness, these expressions for the elements of $\mathrm{Hom}(G_K, P/P^2)$ are not unique.

Proposition 4.2 *An element $\gamma \in H^2(G_K, F^*)$ belongs to Δ_{C_2} if and only if*

- *$\overline{\gamma} = (t, b)_P$, with $t \in K \setminus K^2$ and $b \in F^*$ is such that $F(\sqrt{b})$ is isomorphic to a maximal subfield of $\mathcal{B}$.*
- *$\gamma_\pm = (t, \mathrm{sign}(b))_K$.*

Proof: Let ψ be a morphism whose image is isomorphic to C_2. Then the fixed field of $\ker\psi$ is $K(\sqrt{t})$ for some $t \in K \setminus K^2$, and ψ is the inflation of a morphism (that we also call ψ) from $\mathrm{Gal}(K(\sqrt{t})/K)$, which is determined by the image of a generator σ of the Galois group. If $\psi(\sigma) = \overline{y}$ (here $\overline{y}$ means the class of y in $\mathcal{B}^*/F^*$), then $y^2 = b \in F^*$ and $y \notin F^*$. That is, $F(\sqrt{b})$ is isomorphic to a maximal subfield of $\mathcal{B}$. From the explicit description of δ given in (6), a straightforward computation shows that a cocycle c representing $\delta(\psi)$ is given by

$$c(1,1) = c(1,\sigma) = c(\sigma,1) = 1, \quad c(\sigma,\sigma) = b.$$

By taking the sign of this cocycle we obtain a representant for $\delta(\psi)_\pm$, and it corresponds to the quaternion algebra $(t, \mathrm{sign}(b))_K$. The cocycle c^2 is the coboundary of the map $1 \mapsto 1$, $\sigma \mapsto b$, and by 2.2 the component $\overline{\delta(\psi)}$ is $(t, b)_P$.

Now, for $t \in K \setminus K^2$ and $b \in F^*$ such that $F(\sqrt{b})$ is isomorphic to a maximal subfield of $\mathcal{B}$, take $y \in \mathcal{B}$ with $y^2 = b$. Then the morphism ψ: $\mathrm{Gal}(K(\sqrt{t})/K) \to \mathcal{B}^*/F^*$ that sends a generator σ to $\overline{y}$ has image isomorphic to C_2, and by the previous argument the components of $\delta(\psi)$ are $\delta(\psi)_\pm = (t, \mathrm{sign}(b))_K$ and $\overline{\delta(\psi)} = (t, b)_P$. $\square$

Proposition 4.3 *An element $\gamma \in H^2(G_K, F^*)$ lies in $\Delta_{C_2 \times C_2}$ if and only if*

- $\overline{\gamma} = (s, a)_P \cdot (t, b)_P$, *where $s, t \in K \setminus K^2$ and $\mathcal{B} \simeq (a, b)_F$ with a positive.*
- $\gamma_\pm = (\mathrm{sign}(b)s, t)_K$.

Proof: If ψ is a morphism with image isomorphic to $C_2 \times C_2$, it factorizes through a finite Galois extension M/K with $\mathrm{Gal}(M/K) \simeq C_2 \times C_2$. We write M as $M = K(\sqrt{s}, \sqrt{t})$, and let σ, τ be the generators of the Galois group such that $M^{\langle\sigma\rangle} = K(\sqrt{t})$ and $M^{\langle\tau\rangle} = K(\sqrt{s})$. If $\overline{x} = \psi(\sigma)$ and $\overline{y} = \psi(\tau)$, we know that $x^2 = a \in F^*$, $y^2 = b \in F^*$ and $xy = \varepsilon yx$ for some $\varepsilon \in F^*$. In fact, multiplying this expression on the left by x we see that necessarily $\varepsilon = -1$, and hence $\mathcal{B} \simeq (a, b)_F$.

Let $\gamma_{s,a}$ be the cocycle in $Z^2(\mathrm{Gal}(M/K), F^*)$ defined as the inflation of the cocycle

$$\gamma_{s,a}(1,1) = \gamma_{s,a}(\sigma,1) = \gamma_{s,a}(1,\sigma) = 1, \quad \gamma_{s,a}(\sigma,\sigma) = a,$$

and in a similar way we define the cocycle $\gamma_{t,b}$ by means of

$$\gamma_{t,b}(1,1) = \gamma_{t,b}(\tau,1) = \gamma_{t,b}(1,\tau) = 1, \quad \gamma_{t,b}(\tau,\tau) = b.$$

Let χ_s and χ_t be the elements in $\mathrm{Hom}(\mathrm{Gal}(M/K), \mathbb{Z}/2\mathbb{Z})$ defined by ${}^\rho\sqrt{s}/\sqrt{s} = (-1)^{\chi_s(\rho)}$ and ${}^\rho\sqrt{t}/\sqrt{t} = (-1)^{\chi_t(\rho)}$, and let $\gamma_{s,t}$ be the 2-cocycle in $Z^2(\mathrm{Gal}(M/K), \{\pm 1\})$ defined by $\gamma_{s,t}(\rho, \mu) = (-1)^{\chi_s(\mu)\chi_t(\rho)}$. Then, a direct computation gives that a cocycle representing $\delta(\psi)$ is the product of these three 2-cocycles: $c = \gamma_{s,t} \cdot \gamma_{s,a} \cdot \gamma_{t,b}$. It is well known that $\gamma_{s,t}$ represents the quaternion algebra $(s, t)_K$, and then we have that $\delta(\psi)_\pm = (s, t)_K \cdot (s, \mathrm{sign}(a))_K \cdot (t, \mathrm{sign}(b))_K$. Since $\mathcal{B}$ is totally indefinite, we can suppose that a is positive, and then $\delta(\psi)_\pm = (\mathrm{sign}(b)\, s, t)_K$. Arguing as in the proof of 4.2, the component $\overline{\delta(\psi)}$ is easily seen to be $(s, a)_P \cdot (t, b)_P$.

Finally, suppose that $\mathcal{B} \simeq (a, b)_F$ where the element a is positive. Let s, t be in $K \setminus K^2$, and let $x, y \in \mathcal{B}$ be such that $x^2 = a$, $y^2 = b$ and $xy = -yx$. With the same notations as before for $\mathrm{Gal}(K(\sqrt{s}, \sqrt{t})/K)$, the map ψ that sends σ to $\overline{x}$ and τ to $\overline{y}$ satisfies that $\delta(\psi)_\pm = (\mathrm{sign}(b)\, s, t)_K$ and $\overline{\delta(\psi)} = (s, a)_P \cdot (t, b)_P$.

$\square$

Proposition 4.4 *Suppose that $\mathcal{B}^*/F^*$ contains a subgroup isomorphic to C_n for some $n > 2$, and let ζ_n be a primitive n-th root of unity in $\overline{F}$ and $\alpha = 2 + \zeta_n + \zeta_n^{-1}$. An element $\gamma \in H^2(G_K, F^*)$ lies in Δ_{C_n} if, and only if, there exists a cyclic extension M/K, with $\mathrm{Gal}(M/K) = \langle\sigma\rangle$ such that*

- $\overline{\gamma} = (t, \alpha)$, *where $M(\sqrt{t}) = M^{\langle\sigma^2\rangle}$.*
- $\gamma_\pm$ *is represented by the cocycle*

$$c_\pm(\sigma^i, \sigma^j) = \begin{cases} 1 & \text{if } i + j < n, \\ -1 & \text{if } i + j \geqslant n, \end{cases} \tag{7}$$

We note that if n is odd then $\Delta_{C_n} = \{1\}$.

Proof: Let ψ be a morphism with image isomorphic to C_n. Then the fixed field for $\ker\psi$ is a cyclic extension M/K with $\mathrm{Gal}(M/K) = \langle\sigma\rangle$. The element $x \in \mathcal{B}^*$ such that $\psi(\sigma) = \overline{x}$ has the property that $a = x^n$ lies in F^*. Since $\psi(\sigma^i) = \overline{x^i}$, a straightforward computation shows that $\delta(\psi)$ is given by

$$c(\sigma^i, \sigma^j) = \begin{cases} 1 & \text{if } i+j < n, \\ a & \text{if } i+j \geqslant n. \end{cases} \tag{8}$$

By [1, Lemma 2.1] we can suppose that $x = 1+\zeta$ with $\zeta \in \mathcal{B}^*$ an element of order n. We identify ζ with ζ_n and then by 4.1 we see that $\zeta + \zeta^{-1} \in F^*$. From $(1+\zeta)^2\zeta^{-1} = 2+\zeta+\zeta^{-1}$ we see that $(1+\zeta)^{2n} = (2+\zeta+\zeta^{-1})^n$, and if we define $\alpha = (2+\zeta+\zeta^{-1}) \in F^*$ we have that $a^2 = x^{2n} = (1+\zeta)^{2n} = \alpha^n$. Therefore, the cocycle c^2 is the coboundary of the map $\sigma^i \mapsto \alpha^i$, $0 \leqslant i < n$, and by 2.2 the component $\overline{\delta(\psi)}$ is the map that sends σ to the class of α in P/P^2. Clearly σ^2 is in the kernel of this map, and since $\langle\sigma\rangle = \langle\sigma^2\rangle$ if n is odd, then $\overline{\delta(\psi)}$ is trivial in this case, while if n is even and $K(\sqrt{t})$ is the fixed field of M by $\langle\sigma^2\rangle$, then $\overline{\delta(\psi)} = (t,\alpha)_P$.

A cocycle representing $\delta(\psi)_\pm$ is the sign of (8). If n is odd, the cohomology class of this cocycle is always trivial (it is the coboundary of the map $\sigma^i \mapsto (\operatorname{sign} a)^i$ for $0 \leqslant i < n$). If n is even then a is a totally negative element, because

$$a = x^n = (1+\zeta)^n = (2+\zeta+\zeta^{-1})^{n/2}\zeta^{n/2} = -(2+\zeta+\zeta^{-1})^{n/2},$$

and $2+\zeta+\zeta^{-1}$ is positive due to the identification of ζ with ζ_n. This gives that $\delta(\psi)_\pm$ is given by (7).

Finally, if t, M, σ and α are as in the statement of the proposition, the map ψ sending σ to $\overline{(1+\zeta)}$ with $\zeta \in B^*$ an element of order n gives a $\delta(\psi)$ with the predicted components. ☐

Proposition 4.5 *Suppose that $\mathcal{B}^*/F^*$ contains a subgroup isomorphic to D_{2n} for some $n > 2$. Let ζ_n be a primitive n-th root of unity in $\overline{F}$, $\alpha = 2+\zeta_n+\zeta_n^{-1}$ and $d = (\zeta_n+\zeta_n^{-1})^2 - 4$. A cohomology class $\gamma \in H^2(G_K, F^*)$ lies in $\Delta_{D_{2n}}$ if, and only if, there exists a dihedral extension M/K, with $\mathrm{Gal}(M/K) = \langle\sigma,\tau \mid \sigma^n = 1, \tau^2 = 1, \sigma\tau = \tau\sigma^{-1}\rangle$ such that*

- *$\overline{\gamma} = (s,\alpha)_P \cdot (t,b)_P$, where $L(\sqrt{s}) = M^{\langle\sigma^2,\tau\rangle}$, $L(\sqrt{t}) = M^{\langle\sigma\rangle}$ and $b \in F^*$ satisfies that $\mathcal{B} \simeq (d,b)_F$.*
- *$\gamma_\pm$ is given by the cocycle*

$$c_\pm(\sigma^i\tau, \sigma^{i'}\tau^{j'}) = \begin{cases} 1 & \text{if } \ i-i' \geqslant 0 \\ -1 & \text{if } \ i-i' < 0, \end{cases} \qquad c_\pm(\sigma^i, \sigma^{i'}\tau^{j'}) = \begin{cases} 1 & \text{if } \ i+i' < n \\ -1 & \text{if } \ i+i' \geqslant n, \end{cases} \tag{9}$$

We note that if n is odd, then $\overline{\gamma} = (t,b)_P$ and $\gamma_\pm = 1$.

Proof: Let ψ be a morphism with image isomorphic to D_{2n}. It factorizes through a dihedral extension M with $\mathrm{Gal}(M/K) = \langle\sigma,\tau\rangle$ and the relations between the generators as in the proposition. If we call $\overline{x} = \psi(\sigma)$, $\overline{y} = \psi(\tau)$, we know that $x^n = a \in F^*$, $y^2 = b \in F^*$ and there exist some $\varepsilon \in F^*$ such that $xy = \varepsilon yx^{-1}$. Multiplying in the left by x^{n-1} we find that $x^ny = \varepsilon^n yx^{-n}$ and hence $\varepsilon^n = a^2$. Now we show that, in fact, ε can be identified with α. Indeed,

$x = 1+\zeta$ with $\zeta \in \mathcal{B}^*$ of order n that we identify with ζ_n, and so $x^{-1} = (1+\zeta^{-1})(2+\zeta+\zeta^{-1})^{-1}$. Since $F(\zeta)$ is a maximal subfield of $\mathcal{B}$ different from $F(y)$, the conjugation by y is a non-trivial automorphism of $F(\zeta)/F$. The only such automorphism is complex conjugation, which sends ζ to ζ^{-1}, and therefore $y^{-1}\zeta y = \zeta^{-1}$. This implies that $(1+\zeta)y = y(1+\zeta^{-1})$, and this is $xy = (2+\zeta+\zeta^{-1})yx^{-1}$ which proves that $\varepsilon = (2+\zeta+\zeta^{-1})$, which is identified with α.

In order to give a compact expression for $\delta(\psi)$ we first define a cocycle γ_b:

$$\gamma_b(\sigma^i\tau^j, \sigma^{i'}\tau^{j'}) = \begin{cases} 1 & \text{if } j+j' < 2, \\ b & \text{if } j+j' = 2. \end{cases}$$

and a cocycle e:

$$e(\sigma^i\tau, \sigma^{i'}\tau^{j'}) = \begin{cases} \alpha^{i'} & \text{if } i-i' \geqslant 0 \\ \alpha^{i'}a^{-1} & \text{if } i-i' < 0, \end{cases} \qquad e(\sigma^i, \sigma^{i'}\tau^{j'}) = \begin{cases} 1 & \text{if } i+i' < n \\ a & \text{if } i+i' \geqslant n. \end{cases} \tag{10}$$

To compute a cocycle that represents $\delta(\psi)$, we take the lift $\tilde{\psi}$ form $\mathcal{B}^*/F^*$ to $\mathcal{B}$ given by $\tilde{\psi}(\sigma^i\tau^j) = x^iy^j$ for $0 \leqslant i < n$, $0 \leqslant j < 2$. Then we have that

$$\begin{aligned}
(\delta(\psi))(\sigma^i\tau, \sigma^{i'}\tau^{j'}) &= \tilde{\psi}(\sigma^i\tau)\tilde{\psi}(\sigma^{i'}\tau^{j'})\tilde{\psi}(\sigma^i\tau\sigma^{i'}\tau^{j'})^{-1} = \tilde{\psi}(\sigma^i\tau)\tilde{\psi}(\sigma^{i'}\tau^{j'})\tilde{\psi}(\sigma^{i-i'}\tau^{1+j'})^{-1} = \\
&= \begin{cases} x^iyx^{i'}y^{j'}(x^{i-i'}y^{(1+j') \bmod 2})^{-1} & \text{if } i-i' \geqslant 0 \\ x^iyx^{i'}y^{j'}(x^{n+(i-i')}y^{(1+j') \bmod 2})^{-1} & \text{if } i-i' < 0 \end{cases} \\
&= \begin{cases} \alpha^{i'}x^{i-i'}y^{1+j'}y^{-(1+j') \bmod 2}\ x^{-(i-i')} & \text{if } i-i' \geqslant 0 \\ \alpha^{i'}x^{i-i'}y^{1+j'}y^{-(1+j') \bmod 2}\ x^{-(i-i')}x^{-n} & \text{if } i-i' < 0 \end{cases} \\
&= \begin{cases} \gamma_b(\sigma^i\tau, \sigma^{i'}\tau^{j'})\alpha^{i'} & \text{if } i-i' \geqslant 0 \\ \gamma_b(\sigma^i\tau, \sigma^{i'}\tau^{j'})\alpha^{i'}a^{-1} & \text{if } i-i' < 0. \end{cases}
\end{aligned}$$

$$\begin{aligned}
(\delta(\psi))(\sigma^i, \sigma^{i'}\tau^{j'}) &= \tilde{\psi}(\sigma^i)\tilde{\psi}(\sigma^{i'}\tau^{j'})\tilde{\psi}(\sigma^i\sigma^{i'}\tau^{j'})^{-1} = \tilde{\psi}(\sigma^i)\tilde{\psi}(\sigma^{i'}\tau^{j'})\tilde{\psi}(\sigma^{i+i'}\tau^{j'})^{-1} = \\
&= \begin{cases} x^ix^{i'}y^{j'}(x^{i+i'}y^{j'})^{-1} & \text{if } i+i' < n \\ x^ix^{i'}y^{j'}(x^{(i+i')-n}y^{j'})^{-1} & \text{if } i+i' \geqslant n \end{cases} \\
&= \begin{cases} x^{i+i'}y^{j'}y^{-j'}x^{-(i+i')} & \text{if } i+i' < n \\ x^{i+i'}y^{j'}y^{-j'}x^{-(i+i')}x^n & \text{if } i+i' \geqslant n \end{cases} \\
&= \begin{cases} \gamma_b(\sigma^i, \sigma^{i'}\tau^{j'}) & \text{if } i+i' < n \\ \gamma_b(\sigma^i, \sigma^{i'}\tau^{j}) \cdot a & \text{if } i+i' \geqslant n. \end{cases}
\end{aligned}$$

From these expressions we see that $\delta(\psi)$ is represented by the cocycle $\gamma_b \cdot e$. Clearly γ_b is 2-torsion since γ_b^2 is the coboundary of the map $d_\gamma(\sigma^i) = 1$, $d_\gamma(\sigma^i\tau) = b$. The cocycle e is 2-torsion as well, and a coboundary for e^2 is given by the map $d_e(\sigma^i\tau^j) = \alpha^i$. If we view d_γ and d_e as taking values in P/P^2, then by 2.2 we have that $\overline{\delta(\psi)}$ is the map $d_e \cdot d_\gamma$. Note that $\langle\sigma^2, \tau\rangle \subseteq \ker d_e$. If n is odd, then $\langle\sigma^2, \tau\rangle = \mathrm{Gal}(M/K)$ and the only contribution to $\overline{\delta(\psi)}$ comes from d_γ, and it is the map $(t, b)_P$. If n is even, then the contribution from γ_e is (s, α), and in this case $\overline{\delta(\psi)} = (s, \alpha)_P \cdot (t, b)_P$.

The component $\delta(\psi)_\pm$ comes from taking the sign in the cocycle $\gamma_b \cdot e$. The element b is totally positive, since by [1, Lemma 2.3] we have that $\mathcal{B} \simeq (d,b)_F$, and d is totally negative. To determine the sign of a, note that from $\alpha^n = a^2$, we have that if n is even then $\alpha^{n/2} = \pm a$. The case $\alpha^{n/2} = a$ is not possible since otherwise $F(x^{n/2}, y)$ would be a subfield of $\mathcal{B}$ of dimension 4 over F. Then $\alpha^{n/2} = -a$ and the fact that α is totally positive forces a to be negative. This gives that $\delta(\psi)_\pm$ is represented by the cocycle (9). If n is odd then $c_\pm$ is the coboundary of the map $\sigma^i\tau^j \mapsto (-1)^i$.

As usual, given an extension M/K, elements $b \in F^*$, $s,t \in K^*$ and $c_\pm \in Z^2(\mathrm{Gal}(M/K), \{\pm 1\})$ with the properties described in the proposition, one can construct easily a map ψ with the prescribed $\delta(\psi)$ just defining $\psi(\sigma) = \overline{x}$ and $\psi(\tau) = \overline{y}$, where $\overline{x}, \overline{y}$ generate a subgroup of $\mathcal{B}^*$ isomorphic to D_{2n} and $y^2 = b$. $\square$

5 Descending the field of definition of the variety

Let B be a building block and let $\gamma = [c_B]$ be its associated cohomology class. Suppose that K is a minimal polyquadratic field of definition of B and of all its endomorphisms. As we have seen, it might exist a variety B_0 defined over a subfield L of K that is isogenous to B, but in this case with $\mathrm{End}^0_L(B_0) \subsetneq \mathrm{End}^0(B_0)$. An interesting case of this situation is when the endomorphisms of B_0 are defined over K.

Lemma 5.1 *Let B be a building block such that B and its endomorphisms are defined up to isogeny over a minimal polyquadratic field K. If B_0 is a variety defined over $L \subsetneq K$ that is isogenous to B, and has all of its endomorphisms defined over K, then* $\mathrm{Gal}(K/L) \simeq C_2$ *or* $\mathrm{Gal}(K/L) \simeq C_2 \times C_2$.

Proof: Put $\mathcal{B} = \mathrm{End}^0(B_0)$, and $F = Z(\mathcal{B})$. Since B_0 is a building block, the elements of F are isogenies defined over L (this is a consequence of the existence of compatible isogenies and the fact that F is commutative). Then the action of G_L in $\mathrm{End}^0(B_0)$ gives an injection $\mathrm{Gal}(K/L) \hookrightarrow \mathrm{Aut}_F(\mathcal{B})$. By the Skolem-Noehter theorem $\mathrm{Aut}_F(\mathcal{B}) \simeq \mathcal{B}^*/F^*$, but $\mathrm{Gal}(K/L)$ is a 2-group, and the only subgroups of $\mathcal{B}^*/F^*$ that are 2-groups are isomorphic to C_2 or to $C_2 \times C_2$. $\square$

Proposition 5.2 *Let B be as in the previous lemma, with associated cohomology class $\gamma = [c_B]$. If L is a quadratic subextension of K, then there exists a variety B_0 defined over L isogenous to B and with $\mathrm{End}^0(B_0) = \mathrm{End}^0_K(B_0)$ if and only if there exists a morphism $\psi\colon G_L \to \mathcal{B}^*/F^*$ with image isomorphic to C_2 such that $\mathrm{Res}^L_{\mathbb{Q}}(\gamma) \cdot \delta(\psi) = 1$.*

If L be a biquadratic subextension of K, then there exists a variety B_0 defined over L isogenous to B and with $\mathrm{End}^0(B_0) = \mathrm{End}^0_K(B_0)$ if and only if there exists a morphism $\psi\colon G_L \to \mathcal{B}^*/F^*$ with image isomorphic to $C_2 \times C_2$ such that $\mathrm{Res}^L_{\mathbb{Q}}(\gamma) \cdot \delta(\psi) = 1$.

Proof: Being the two cases similar, we only prove the biquadratic one because it is the most involved. First of all, suppose that there exists a ψ with image $C_2 \times C_2$ such that $\mathrm{Res}^L_{\mathbb{Q}}(\gamma) \cdot \delta(\psi) = 1$. Write $K = \mathbb{Q}(\sqrt{t_1}, \sqrt{t_2}, \dots, \sqrt{t_n})$ with $K = L(\sqrt{t_1}, \sqrt{t_2})$. We know that $\overline{\gamma} = (t_1, d_1)_P \cdots (t_n, d_n)_P$ for some $d_i \in F^*$, and then $\mathrm{Res}^L_{\mathbb{Q}}(\overline{\gamma}) = (t_1, d_1)_P \cdot (t_2, d_2)_P$. Since $\overline{\delta(\psi)} = \mathrm{Res}^L_{\mathbb{Q}}(\overline{\gamma}) = (t_1, d_1)_P \cdot (t_2, d_2)_P$, as we have seen in the last paragraph of the proof of 4.3, we can assume that ψ is the inflation of a morphism defined in $\mathrm{Gal}(K/\mathbb{Q})$. In particular, we can

suppose that $G_K \subseteq \ker\psi$. For each $\sigma \in G_L$ consider a $\psi_\sigma \in \mathcal{B}^*$ such that $\psi(\sigma) = \psi_\sigma F^*$. Then, the fact that $\mathrm{Res}^L_{\mathbb{Q}}(\gamma)\cdot\delta(\psi) = 1$ translates into the existence of elements $\{\lambda_\sigma \in F^*\}_{\sigma\in G_L}$ such that

$$\mu_\sigma \circ {}^\sigma\mu_\tau \circ \mu_{\sigma\tau}^{-1} \circ \psi_\sigma \circ \psi_\tau \circ \psi_{\sigma\tau}^{-1} = \lambda_\sigma \circ \lambda_\tau \circ \lambda_{\sigma\tau}^{-1}, \quad \text{for all } \sigma,\tau \in G_L,$$

where as usual μ_σ stands for a compatible isogeny $\mu_\sigma\colon {}^\sigma B \to B$. Since λ_σ belongs to the center of $\mathcal{B}$, the isogeny $\lambda_\sigma^{-1}\circ\mu_\sigma$ is again compatible. Hence, changing μ_σ by $\lambda_\sigma^{-1}\circ\mu_\sigma$ we can suppose that

$$\mu_\sigma \circ {}^\sigma\mu_\tau \circ \mu_{\sigma\tau}^{-1} \circ \psi_\sigma \circ \psi_\tau \circ \psi_{\sigma\tau}^{-1} = 1, \quad \text{for all } \sigma,\tau \in G_L. \tag{11}$$

Now, for each $\sigma \in G_L$ we define $\nu_\sigma = \psi_\sigma \circ \mu_\sigma \in \mathrm{Isog}^0({}^\sigma B, B)$. Using (11) it is easy to check that $\nu_\sigma \circ {}^\sigma\nu_\tau = \nu_{\sigma\tau}$ for all $\sigma,\tau \in G_L$. Therefore, by 2.3 there exists a variety B_0 defined over L and an isogeny $\kappa\colon B \to B_0$ such that $\kappa^{-1}\circ{}^\sigma\kappa = \nu_\sigma = \psi_\sigma\circ\mu_\sigma$. Now we have to prove that B_0 has all of its endomorphisms defined over K. Every endomorphism of B_0 can be written as $\kappa\circ\varphi\circ\kappa^{-1}$ for some $\varphi \in \mathrm{End}^0(B)$. Then, for $\sigma \in G_K$ we have that

$$\begin{aligned} {}^\sigma(\kappa\circ\varphi\circ\kappa^{-1}) &= {}^\sigma\kappa\circ{}^\sigma\varphi\circ{}^\sigma\kappa^{-1} = \kappa\circ\psi_\sigma\circ\mu_\sigma\circ{}^\sigma\varphi\circ\mu_\sigma^{-1}\circ\psi_\sigma^{-1}\circ\kappa^{-1} \\ &= \kappa\circ\psi_\sigma\circ\varphi\circ\psi_\sigma^{-1}\circ\kappa^{-1} = \kappa\circ\varphi\circ\kappa^{-1}, \end{aligned}$$

where in the last equality have used that $\psi_\sigma \in F^*$, because $G_K \subseteq \ker\psi$.

For the other implication, suppose that there exists a B_0 defined over L, with all of its endomorphisms defined over K and with an isogeny $\kappa\colon B \to B_0$. For $\sigma \in G_L$ we define $\nu_\sigma = \kappa^{-1}\circ{}^\sigma\kappa \in \mathrm{Isog}^0({}^\sigma B, B)$, and $\psi_\sigma = \nu_\sigma\circ\mu_\sigma^{-1} \in \mathrm{End}^0(B)^* = \mathcal{B}^*$. Since $\nu_\sigma\circ{}^\sigma\nu_\tau\circ\nu_{\sigma\tau}^{-1} = 1$ for every $\sigma,\tau \in G_L$, we have that $\mu_\sigma\circ{}^\sigma\mu_\tau\circ\mu_{\sigma\tau}^{-1}\circ\psi_\sigma\circ\psi_\tau\circ\psi_{\sigma\tau}^{-1} = 1$ for all $\sigma,\tau \in G_L$. Hence, the map $\sigma \mapsto \psi_\sigma F^*\colon G_L \to \mathcal{B}^*/F^*$ is a morphism ψ such that $\mathrm{Res}^L_{\mathbb{Q}}([c_B])\cdot\delta(\psi) = 1$. Moreover, with the same reasoning we made in the first part of the proof, the fact that all the endomorphisms of B_0 are all defined over K implies that $\psi_\sigma \in Z(\mathcal{B}) = F^*$ for all $\sigma \in G_K$. Therefore ψ factorizes through $\mathrm{Gal}(K/L)$, and does not factorize through any subextension K' of K, since otherwise all the endomorphisms of B_0 would be defined over K', contradicting the minimality of K. Therefore, $\mathrm{im}(\psi) \simeq \mathrm{Gal}(K/L) \simeq C_2 \times C_2$. □

6 Examples

In this section we illustrate with some examples the use of the techniques developed so far in studying the field of definition of building blocks up to isogeny. We will use the information provided by the building block table of [4, Section 5.1 of the Appendix]. These data can also be obtained directly by means of the `Magma` functions implemented by Jordi Quer, which are based on the packages of William Stein for modular abelian varieties.

Example. Let B be the only building block of dimension 2 with quaternionic multiplication that is associated to a newform f of level $N = 243$ and trivial Nebentypus, and let $\gamma = [c_B]$ be its cohomology class. The components of γ are $\gamma_\pm = 1$ and $\overline{\gamma} = (-3,6)_P$, and $K_P = \mathbb{Q}(\sqrt{-3})$ is a minimum field of definition of B and of its endomorphisms up to isogeny. The dimension of B is 2, as it is the dimension of A_f; therefore, we know a priori that $\mathbb{Q}$ is a field of definition of B up to isogeny. Let us see now how this can also be deduced using our results. The endomorphism algebra $\mathcal{B}$ is the quaternion algebra over $\mathbb{Q}$ ramified at the primes 2 and 3. The field $\mathbb{Q}(\sqrt{6})$ is isomorphic to a maximal subfield of $\mathcal{B}$, and by 4.2 there exists a morphism $\psi\colon G_{\mathbb{Q}} \to \mathcal{B}^*/\mathbb{Q}^*$

such that $\overline{\delta(\psi)} = (-3,6)_P$ and $\delta(\psi)_\pm = (-3,1)_{\mathbb{Q}}$ which is trivial in $H^2(G_{\mathbb{Q}}, \{\pm 1\})$. Therefore $\gamma \cdot \delta(\psi) = 1$ and we deduce the existence of an abelian variety defined over $\mathbb{Q}$ and isogenous to B.

Example. Let B be the only quaternionic building block of dimension 2 associated to a modular form f of level $N = 60$ with Nebentypus of order 4. In this case the variety A_f is 4-dimensional and the cohomology class associated to B has components $\overline{\gamma} = (5,2)_P \cdot (-3,5)_P$, and $\gamma_\pm$ the quaternion algebra over $\mathbb{Q}$ ramified at the primes 3 and 5. The field $K_P = \mathbb{Q}(\sqrt{5}, \sqrt{-3})$ is the minimum field of definition of the variety and of its endomorphisms up to isogeny, and the algebra $\mathcal{B} = \mathrm{End}^0(B)$ is the quaternion algebra over $\mathbb{Q}$ ramified at 2 and 5, which is isomorphic to $(-2,5)_{\mathbb{Q}}$. Hence, by 4.3 there exists a $\psi\colon G_{\mathbb{Q}} \to \mathcal{B}^*/\mathbb{Q}^*$ such that $\overline{\delta(\psi)} = (5,-2)_P \cdot (-3,5)_P$ and $\delta(\psi)_\pm = (5,3)_{\mathbb{Q}}$, which is the quaternion algebra ramified at 3 and 5. Hence $\gamma \cdot \delta(\psi) = 1$ and by 5.2 there exists a variety B_0 defined over $\mathbb{Q}$ and with all its endomorphisms defined over K_P that is isogenous to B.

Example. Let B be the only quaternionic building block of dimension 2 associated to a newform f of level $N = 80$ and Nebentypus of order 4. Now $\overline{\gamma} = (5,2)_P \cdot (-4,3)_P$ and $\gamma_\pm$ is the quaternion algebra over $\mathbb{Q}$ ramified at 2 and 5. Again K_P, which in this case is $\mathbb{Q}(\sqrt{5}, \sqrt{-1})$, is the minimum field of definition of B and of its endomorphisms up to isogeny.

First, we observe that there does not exist a variety B_0 defined over $\mathbb{Q}$ and with all its endomorphisms defined over K_P. By 5.2 the existence of such variety would be equivalent to the existence of a $\psi\colon G_{\mathbb{Q}} \to \mathcal{B}^*/\mathbb{Q}^*$ with image isomorphic to $C_2 \times C_2$ such that $\overline{\delta(\psi)} = \overline{\gamma}$ and $\delta(\psi)_\pm = \gamma_\pm$. By 4.3, $\overline{\delta(\psi)} = (s,a)_P \cdot (t,b)_P$ with $\mathcal{B} \simeq (a,b)_{\mathbb{Q}}$. If we want $\overline{\delta(\psi)} = \overline{\gamma}$, the only possibilities for a, b modulo squares are the following: $a = 2$ and $b = 3$, $a = 2$ and $b = -3$, $a = -2$ and $b = 3$ or $a = -2$ and $b = -3$. Since $\mathcal{B}$ is the quaternion algebra of discriminant 6, only the first two options are possible. But if $\overline{\delta(\psi)} = (5,2)_P \cdot (-4,3)_P$, from 4.3 we see that $\delta(\psi)_\pm = (5,-4)_{\mathbb{Q}}$, which is not equal to $\gamma_\pm$, and if $\overline{\delta(\psi)} = (5,2)_P \cdot (-4,-3)_P$ then $\delta(\psi)_\pm = (-5,-4)_{\mathbb{Q}}$ which is also not equal to $\gamma_\pm$. Hence there does not exist such a ψ.

Now we will see that there exists a $\psi\colon G_{\mathbb{Q}} \to \mathcal{B}^*/F^*$ with image isomorphic to $D_{2\cdot 4}$ such that $\gamma \cdot \delta(\psi) = 1$. This will tell us that there exists an abelian variety B_0 defined over $\mathbb{Q}$ that is isogenous to B, but that does not have all its endomorphisms defined over K_P. First of all, we observe that $\mathcal{B} \simeq (-1,3)_{\mathbb{Q}}$, and so $\mathcal{B}$ contains a maximal subfield isomorphic to $\mathbb{Q}(i)$, where $i = \sqrt{-1}$. This implies that $\mathcal{B}^*/\mathbb{Q}^*$ contains subgroups isomorphic to $D_{2\cdot 4}$. More precisely, if x, y are elements in $\mathcal{B}$ such that $x^2 = -1$, $y^2 = 3$, and $xy = -yx$, then the subgroup of $\mathcal{B}^*/\mathbb{Q}^*$ generated by $\overline{1+x}$ and $\overline{y}$ is isomorphic to $D_{2\cdot 4}$.

The number field $M = \mathbb{Q}(\sqrt[4]{5}, i)$ has $\mathrm{Gal}(M/\mathbb{Q}) \simeq D_{2\cdot 4}$, generated by the automorphisms $\sigma\colon \sqrt[4]{5} \mapsto i\sqrt[4]{5},\ i \mapsto i$ and $\tau\colon \sqrt[4]{5} \mapsto \sqrt[4]{5},\ i \mapsto -i$. We define $\psi\colon G_{\mathbb{Q}} \to \mathcal{B}^*/F^*$ as the morphism sending σ to $\overline{1+x}$ and τ to $\overline{y}$. From the expressions given in 4.5 we see that $\overline{\delta(\psi)} = (-1,3)_P \cdot (5,2)_P$, which is equal to $\overline{\gamma}$. It only remains to see that $\delta(\psi)_\pm = \gamma_\pm$. Let D be the quaternion algebra associated to $\delta(\psi)_\pm$. Since $\delta(\psi)_\pm \in Z^2(\mathrm{Gal}(M/\mathbb{Q}), \{\pm 1\})$ and the extension $M/\mathbb{Q}$ only ramifies at the primes 2 and 5, D can only ramify at the places 2, 5 and ∞ (see [2, Proposition 18.5]). We will see that $D \otimes_{\mathbb{Q}} \mathbb{Q}(i)$ is not trivial in the Brauer group (and therefore D ramifies at some prime), and that $D \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{5})$ is trivial (and therefore D does not ramify at ∞). These two conditions imply that D ramifies exactly at 2 and 5.

Since $\mathrm{Gal}(M/\mathbb{Q}(i)) = \langle \sigma \rangle$, a 2-cocycle c representing $D \otimes_{\mathbb{Q}} \mathbb{Q}(i)$ is the restriction to the

subgroup $\langle\sigma\rangle \subseteq \mathrm{Gal}(M/\mathbb{Q})$ of a cocycle representing $\delta(\psi)_\pm$. From (9) we obtain that

$$c(\sigma^i,\sigma^j)=\begin{cases}1 & \text{if } i+j<4\\ -1 & \text{if } i+j\geqslant 4.\end{cases}$$

By [2, Lemma 15.1] the algebra associated to this cocycle is trivial if and only if $-1\in \mathrm{Nm}_{M/\mathbb{Q}(i)}(M)$, where $\mathrm{Nm}_{M/\mathbb{Q}(i)}$ refers to the norm in the extension $M/\mathbb{Q}(i)$. But -1 is not a norm of this extension, hence $D\otimes_{\mathbb{Q}}\mathbb{Q}(i)$ is non-trivial in the Brauer group.

Since $\mathrm{Gal}(M/\mathbb{Q}(\sqrt{5}))=\langle\sigma^2,\tau\rangle$, a 2-cocycle c representing $D\otimes_{\mathbb{Q}}\mathbb{Q}(\sqrt{5})$ is the restriction to $\langle\sigma^2,\tau\rangle\subseteq \mathrm{Gal}(M/\mathbb{Q})$ of a cocycle representing $\delta(\psi)_\pm$. Again from (9) we obtain the following:

$$\begin{array}{llll}
c(1,1)=1 & c(\sigma^2,1)=1 & c(\tau,1)=1 & c(\sigma^2\tau,1)=1\\
c(1,\sigma^2)=1 & c(\sigma^2,\sigma^2)=-1 & c(\tau,\sigma^2)=-1 & c(\sigma^2\tau,\sigma^2)=1\\
c(1,\tau)=1 & c(\sigma^2,\tau)=1 & c(\tau,\tau)=1 & c(\sigma^2\tau,\tau)=1\\
c(1,\sigma^2\tau)=1 & c(\sigma^2,\sigma^2\tau)=-1 & c(\tau,\sigma^2\tau)=-1 & c(\sigma^2\tau,\sigma^2\tau)=1.
\end{array}$$

To see that the cohomology class of this cocycle in $H^2(\mathrm{Gal}(M/\mathbb{Q}(\sqrt{5})),M^*)$ is trivial (where now the action is the natural Galois action), we define a map λ by $\lambda(1)=1$, $\lambda(\sigma^2)=i$, $\lambda(\tau)=i$ and $\lambda(\sigma^2\tau)=-i$. Now a computation shows that $c(\rho,\mu)=\lambda(\rho)\cdot{}^{\rho}\lambda(\mu)\cdot\lambda(\rho\mu)^{-1}$, for all $\rho,\mu\in\mathrm{Gal}(M/\mathbb{Q}(\sqrt{5}))$.

Example. Consider the building block B in the table associated with a newform of conductor 336. For this variety $\overline{\gamma}=(-3,11)_P$ and $\gamma_\pm$ is the quaternion algebra ramified at 2 and 3. Hence $K_P=\mathbb{Q}(\sqrt{-3})$ and since $\mathrm{Res}_{\mathbb{Q}}^{K_P}(\gamma_\pm)=1$ we have that K_P is the minimum field of definition of B and of its endomorphisms up to isogeny. We will show that B is not isogenous to any variety defined over $\mathbb{Q}$.

As K_P is a quadratic number field and $\gamma_\pm\neq 1$, the only morphisms ψ we have to consider are those with image isomorphic to C_2 or to C_n for some even $n>2$. The only such values of n with $\mathcal{B}^*/\mathbb{Q}^*$ containing a subgroup isomorphic to C_n are $n=4$ and $n=6$. Since the component $\overline{\delta(\psi)}$ associated to a ψ with image C_n has the form $(t,2+\zeta_n+\zeta_n^{-1})$, and for $n=4,6$ we have that $2+\zeta_n+\zeta_n^{-1}$ is not congruent to 11 modulo $\{\pm1\}\mathbb{Q}^{*2}$, it turns out that there does not exist any ψ with image C_4 or C_6 such that $\gamma\cdot\delta(\psi)=1$. If ψ has image C_2, the only possibilities are $\overline{\delta(\psi)}=(-3,11)$ or $\overline{\delta(\psi)}=(-3,-11)$. In the first case we would have $\delta(\psi)_\pm=(-3,1)_{\mathbb{Q}}$ and in the second case $\delta(\psi)_\pm=(-3,-1)$. In both cases $\delta(\psi)_\pm\neq\gamma_\pm$, and thus there does not exist a ψ with image C_2 such that $\gamma\cdot\delta(\psi)=1$.

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